Int. J. Solids Structures Vol. 29, No. 1, pp. 57–68, 1992
Printed in Great Britain.

0020-7683/92 \$3.00 + .00

PARAMETRIZED VARIATIONAL PRINCIPLES ENCOMPASSING COMPRESSIBLE AND INCOMPRESSIBLE ELASTICITY

CARLOS A. FELIPPA
Department of Aerospace Engineering Sciences and Center for Space Structures and Controls, University of Colorado, Boulder, CO 80309-0429, U.S.A.

(*Received* 16 *July* 1990)

Abstract—A parametrized five-field variational principle that can accommodate both compressible and incompressible hyperelasticity is presented. The primary variables are mean and deviatoric stresses, mean and deviatoric strains, and displacements. Through appropriate selection of parameters the functional of this general principle specializes to those previously presented by Atluri–Reissner, Herrmann and Franca.

1. GOVERNING EQUATIONS

Consider a *linearly hyperelastic body* under static loading that occupies the volume V. The body is bounded by the surface S, which is decomposed into $S: S_d \cup S_t$. Displacements are prescribed on S_d while surface tractions are prescribed on S_t. The outward unit normal on S is denoted by $\mathbf{n} \equiv n_i$.

The three unknown volume fields are displacements $\mathbf{u} \equiv u_i$, infinitesimal strains $\mathbf{e} \equiv e_{ij}$, and stresses $\boldsymbol{\sigma} \equiv \sigma_{ij}$. The problem data include: the body force field $\mathbf{b} \equiv b_i$ in V, prescribed displacements $\hat{\mathbf{d}} \equiv \hat{d}_i$ on S_d, and prescribed surface tractions $\hat{\mathbf{t}} \equiv \hat{t}_i$ on S_t.

The relations between the volume fields are the strain displacement equations

$$\mathbf{e} = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u}) = \mathbf{D}\mathbf{u} \quad \text{or} \quad e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \text{ in } V, \tag{1}$$

the constitutive equations

$$\boldsymbol{\sigma} = \mathbf{E}\mathbf{e} \quad \text{or} \quad \sigma_{ij} = E_{ijkl}e_{kl} \text{ in } V, \tag{2}$$

and the equilibrium (balance) equations

$$-\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{D}^*\boldsymbol{\sigma} = \mathbf{b} \quad \text{or} \quad \sigma_{ij,j} + b_i = 0 \text{ in } V, \tag{3}$$

in which $\mathbf{D}^* = -\mathbf{div}$ denotes the adjoint operator of the symmetric gradient $\mathbf{D} = \frac{1}{2}(\nabla + \nabla^T)$.

The stress vector with respect to a direction defined by the unit vector $\mathbf{v}$ is denoted as $\boldsymbol{\sigma}_v = \boldsymbol{\sigma}\cdot\mathbf{v}$, or $\sigma_{vi} = \sigma_{ij}v_j$. On S the surface-traction stress vector is defined as $\boldsymbol{\sigma}_n = \boldsymbol{\sigma}\cdot\mathbf{n}$ or $\sigma_{ni} = \sigma_{ij}n_j$. With this notation the traction and displacement boundary conditions may be stated as

$$\sigma_n = \hat{\mathbf{t}}_i \text{ or } \sigma_{ij}n_j = \hat{t}_i \text{ on } S_t, \quad \text{and} \quad \mathbf{u} = \hat{\mathbf{d}} \text{ or } u_i = \hat{d}_i \text{ on } S_d. \tag{4}$$

2. NOTATION

2.1. *Field dependency*

In this investigation of variational methods, the notational conventions in Felippa (1989a,b,c) and Felippa and Militello (1989, 1990) are used. An *independently varied* field will be identified by a superposed tilde, for example $\tilde{\mathbf{u}}$. A dependent field is identified by writing the independent field symbol as superscript. For example, if the displacements are independently varied, the derived strain and stress fields are

$$\tilde{\mathbf{e}}^u = \tfrac{1}{2}(\nabla+\nabla^T)\tilde{\mathbf{u}} = \mathbf{D}\tilde{\mathbf{u}}, \quad \tilde{\boldsymbol{\sigma}}^u = \mathbf{E}\tilde{\mathbf{e}}^u = \mathbf{E}\mathbf{D}\tilde{\mathbf{u}}. \tag{5}$$

Using this convention, tildeless symbols such as $\mathbf{u}$, $\mathbf{e}$ and $\boldsymbol{\sigma}$ are reserved for the *exact* or for *generic* fields.

2.2. *Integral abbreviations*

Volume and surface integrals may be abbreviated by placing domain-subscripted parentheses and square brackets, respectively, around the integrand. For example:

$$(f)_V \stackrel{\text{def}}{=} \int_V f\,\mathrm{d}V, \quad [f]_S \stackrel{\text{def}}{=} \int_S f\,\mathrm{d}S, \quad [f]_{S_d} \stackrel{\text{def}}{=} \int_{S_d} f\,\mathrm{d}S, \quad [f]_{S_t} \stackrel{\text{def}}{=} \int_{S_t} f\,\mathrm{d}S. \tag{6}$$

If $\mathbf{f}$ and $\mathbf{g}$ are vector functions, and $\mathbf{p}$ and $\mathbf{q}$ tensor functions, their inner product over V is denoted in the usual manner:

$$(\mathbf{f},\mathbf{g})_V \stackrel{\text{def}}{=} \int_V f_i g_i\,\mathrm{d}V, \quad (\mathbf{p},\mathbf{q})_V \stackrel{\text{def}}{=} \int_V p_{ij} q_{ij}\,\mathrm{d}V, \tag{7}$$

and similarly for surface integrals, in which case square brackets are used.

2.3. *Stress and strain vectors*

To facilitate the construction of variational matrix expressions, stresses and strains will be arranged as 6-component column vectors constructed from the tensors σ_{ij} and e_{ij} following the usual conventions of structural mechanics:

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}, \quad \mathbf{e} = \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{12} \\ 2e_{23} \\ 2e_{31} \end{Bmatrix}. \tag{8}$$

Then $(\boldsymbol{\sigma},\mathbf{e})_V = (\sigma_{ij}e_{ij})_V = (\boldsymbol{\sigma}^T\mathbf{e})_V$, and so on. Similarly, fourth-order constitutive tensors such as E_{ijkl} are arranged as symmetric 6×6 matrices (resulting from their restriction to the space of symmetric stress–strain tensors) in the usual manner.

3. STRESS STRAIN SPLITTINGS

For incompressible materials, in which $\mathbf{div}\,\mathbf{u} = \mathbf{tr}\,\nabla\mathbf{u} = u_{i,i} = 0$, the stress–strain relation (2) only holds in the space of traceless strain tensors, and its inverse does not exist. With a view to including both compressible and incompressible elasticity in the variational principles, some general splittings of the strain and stress fields are studied below. Define (actual) pressure p and total strain condensation (negative of the volumetric strain) θ as

$$\begin{aligned} p &= -\tfrac{1}{3}\,\mathbf{tr}\,\boldsymbol{\sigma} = -\tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) \\ \theta &= -\mathbf{tr}\,\mathbf{e} = -(e_{11}+e_{22}+e_{33}) = -\mathbf{div}\,\mathbf{u}. \end{aligned} \tag{9}$$

Throughout this paper it shall be assumed that the material is *volumetrically isotropic* in the sense

$$p = k\theta, \tag{10}$$

where $k > 0$ is the modulus of compression (one third of the bulk modulus K). In the incompressible limit, $k \to \infty$.

3.1. *Parametrized splitting*

A family of stress–strain splittings considered here is

$$\sigma_{ij} = s(\xi)_{ij} - \xi p \delta_{ij}, \quad e_{ij} = g(\eta)_{ij} - \tfrac{1}{3}\eta\theta\delta_{ij}, \tag{11}$$

where δ_{ij} is the Kronecker delta, and ξ and η are scalars in the range [0, 1] that determine the splitting. If $\xi = 0$, $s(0)_{ij} \equiv \sigma_{ij}$, whereas if $\xi = 1$, $s(1)_{ij}$ reduce to the usual deviatoric stresses s_{ij} and the argument ξ will be omitted. If $\xi = 0$, $g(0)_{ij} \equiv e_{ij}$, whereas if $\xi = 1$, $g(1)_{ij}$ reduce to the usual deviatoric strains g_{ij} and the argument η will be omitted.

Using the matrix notation (8) for strains and stresses, (11) is represented as

$$\boldsymbol{\sigma} = \mathbf{s}(\xi) - \xi p\mathbf{h}, \quad \mathbf{e} = \mathbf{g}(\eta) - \eta\theta\mathbf{h}, \tag{12}$$

where $\mathbf{h}$ is the 6-component column vector:

$$\mathbf{h} = \{1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0\}^T. \tag{13}$$

Note that $\mathbf{h}^T\mathbf{h} = 3$, $\mathbf{h}^T\boldsymbol{\sigma} = \mathbf{tr}\,\boldsymbol{\sigma} = -3p$, $\mathbf{h}^T\mathbf{e} = \mathbf{tr}\,\mathbf{e} = -\theta$, $\mathbf{h}^T\mathbf{s}(\xi) = \mathbf{tr}\,\mathbf{s}(\xi) = -3(1-\xi)p$, $\mathbf{h}^T\mathbf{g}(\eta) = \mathbf{tr}\,\mathbf{g}(\eta) = -(1-\eta)\theta$, and $\mathbf{h}^T\mathbf{s} = \mathbf{h}^T\mathbf{g} = 0$.

3.2. *Constraints on ξ and η*

Parameters ξ and η are not independent but chosen so that $\mathbf{s}(\xi)$ and $\mathbf{g}(\eta)$ are connected by an invertible "deviatoric" constitutive equation

$$\mathbf{s}(\xi) = \mathbf{C}\mathbf{g}(\eta) \quad \text{or} \quad s(\xi)_{ij} = C_{ijkl}\,g(\eta)_{kl}, \tag{14}$$

where $\mathbf{C}$ is finite and nonsingular. This condition is assumed to hold if $\xi = \eta = 1$ for any material. For other values the choice is possible if the material is fully isotropic because, if this is so, (2) may be written [see e.g. Section 22 of Gurtin (1972)]:

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk} \quad \text{or} \quad \boldsymbol{\sigma} = 2\mu\mathbf{e} - \lambda\theta\mathbf{h}, \tag{15}$$

where μ and λ are the Lamé coefficients (μ is the same as the shear modulus G), so that $\mathbf{C} = 2\mu\mathbf{I}$. Furthermore, μ, λ and k are related to the elastic modulus E and Poisson's ratio ν through

$$k = \frac{\lambda(1+\nu)}{3\nu} = \frac{E}{3(1-2\nu)} = \tfrac{1}{3}(3\lambda+2\mu), \quad \mu = \frac{\lambda(1-2\nu)}{2\nu} = \tfrac{3}{2}(k-\lambda) = \frac{E}{2(1+\nu)}. \tag{16}$$

Substituting these relations into (15) and (14) one obtains the relation

$$(1+\nu)\xi - (1-2\nu)\eta = 3\nu. \tag{17}$$

The pair $\xi = \eta = 1$ satisfies this constraint for any ν. If $\nu \neq 0.5$, specifying $0 \leqslant \xi < 1$ or η determines the other; for example if $\eta = 0$, $\xi = 3\nu/(1+\nu)$. If the material is incompressible, i.e. $\nu = 0.5$, $\xi = 1$ regardless of the value of η.

3.3. *Deviatoric splitting*

The usual deviatoric stress–strain splitting is obtained by taking $\xi = \eta = 1$:

$$\boldsymbol{\sigma} = \mathbf{s} - p\mathbf{h}, \quad \mathbf{e} = \mathbf{g} - \tfrac{1}{3}\theta\mathbf{h}. \tag{18}$$

As noted above, this choice satisfies the condition (14) for isotropic or anisotropic materials.

3.4. *Lamé splitting*

The Lamé splitting for isotropic materials—so called because of its intimate relationship with the constitutive form (15) that displays the two Lamé coefficients—is obtained if $\eta = 0$ so that $\mathbf{g} = \mathbf{e}$. Then ξ is chosen so that $\boldsymbol{\tau} = \mathbf{s}(\xi) = 2\mu\mathbf{e}$:

$$\boldsymbol{\sigma} = \mathbf{Ce} - \xi p\mathbf{h} = 2\mu\mathbf{e} - \frac{3\nu}{1+\nu}p\mathbf{h} = \boldsymbol{\tau} - q\mathbf{h}. \tag{19}$$

In the literature $q = \xi p$ is called the pseudo pressure whereas $\boldsymbol{\tau} = \mathbf{s}(\xi) = 2\mu\mathbf{e} = \mathbf{Ce}$ is called the extra stress, although a better name would be pseudo deviatoric stress. In the incompressible limit, pseudo pressure q and extra stress $\boldsymbol{\tau}$ reduce to ordinary pressure p and deviatoric stress $\mathbf{s}$, respectively.

Although the Lamé splitting may in principle be extended to anisotropic materials, parameter ξ then becomes a matrix: $\mathbf{I} - (3k)^{-1}\mathbf{C}$, which complicates derivations substantially. The same is true of (12) unless $\xi = \eta = 1$. It follows that splittings other than (18) are of limited value for non-isotropic behavior.

4. THE GENERALIZED STRAIN ENERGY

The variational principles of linear elasticity studied here have the general form

$$\Pi = U - P. \tag{20}$$

Here U is the generalized strain energy, which characterizes the stored energy of deformation, and P is the forcing potential, which characterizes all other contributions. The conventional form of P is

$$P^c = (\mathbf{b}, \mathbf{u})_V + [\mathbf{u} - \hat{\mathbf{d}}, \boldsymbol{\sigma}_n]_{S_d} + [\hat{\mathbf{t}}, \mathbf{u}]_{S_t}. \tag{21}$$

Two other forms of P, which are of interest in hybrid finite element formulations, called P^d and P^t for displacement-generalized and traction-generalized, respectively, are studied in Felippa (1989a,b,c) and Felippa and Militello (1989, 1990). As this term is not affected by material behavior, attention will be focused on U.

For a compressible material, the generalized strain energy introduced in Felippa and Militello (1989, 1990) has the following parametrized structure:

$$U = \tfrac{1}{2}j_{11}(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^\sigma)_V + j_{12}(\tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{e}})_V + j_{13}(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u)_V + \tfrac{1}{2}j_{22}(\boldsymbol{\sigma}^e, \tilde{\mathbf{e}})_V + j_{23}(\boldsymbol{\sigma}^e, \mathbf{e}^u)_V + \tfrac{1}{2}j_{33}(\boldsymbol{\sigma}^u, \mathbf{e}^u)_V, \tag{22}$$

where j_{11} through j_{33} are numerical coefficients. The three independent fields are stresses $\tilde{\boldsymbol{\sigma}}$, strains $\tilde{\mathbf{e}}$ and displacements $\tilde{\mathbf{u}}$. Following the notational conventions stated in Section 2, the derived fields that appear in (22) are

$$\boldsymbol{\sigma}^e = \mathbf{E}\tilde{\mathbf{e}}, \quad \boldsymbol{\sigma}^u = \mathbf{ED}\tilde{\mathbf{u}}, \quad \mathbf{e}^\sigma = \mathbf{E}^{-1}\tilde{\boldsymbol{\sigma}}, \quad \mathbf{e}^u = \mathbf{D}\tilde{\mathbf{u}}. \tag{23}$$

As an example, the U of Hu–Washizu's functional is obtained by setting $j_{12} = -1$, $j_{13} = 1$, $j_{22} = 1$, all others being zero:

$$U_H(\tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{e}}, \tilde{\mathbf{u}}) = \tfrac{1}{2}(\boldsymbol{\sigma}^e, \tilde{\mathbf{e}})_V + \tfrac{1}{2}(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u - \tilde{\mathbf{e}})_V + \tfrac{1}{2}(\boldsymbol{\sigma}^u - \boldsymbol{\sigma}^e, \mathbf{e}^\sigma)_V = \tfrac{1}{2}(\boldsymbol{\sigma}^e, \tilde{\mathbf{e}})_V + (\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u - \tilde{\mathbf{e}})_V. \tag{24}$$

Equation (22) can be rewritten in matrix form as

$$U = \frac{1}{2}\int_V \begin{Bmatrix} \tilde{\boldsymbol{\sigma}} \\ \boldsymbol{\sigma}^e \\ \boldsymbol{\sigma}^u \end{Bmatrix}^T \begin{bmatrix} j_{11}\mathbf{I} & j_{12}\mathbf{I} & j_{13}\mathbf{I} \\ & j_{22}\mathbf{I} & j_{23}\mathbf{I} \\ symm & & j_{33}\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{e}^\sigma \\ \tilde{\mathbf{e}} \\ \mathbf{e}^u \end{Bmatrix} \mathrm{d}V \tag{25}$$

where $\mathbf{I}$ denotes the 6×6 identity matrix. The functional-generating symmetric matrix (to justify the symmetry note, for example, that $j_{13}(\tilde{\sigma}, \mathbf{e}^u)_V = \frac{1}{2}j_{13}(\tilde{\sigma}, \mathbf{e}^u)_V + \frac{1}{2}j_{13}(\mathbf{e}^\sigma, \sigma^u)_V$, and so on)

$$\mathbf{J}_3 = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{12} & j_{22} & j_{23} \\ j_{13} & j_{23} & j_{33} \end{bmatrix} \tag{26}$$

is seen to fully characterize (22) hence, once the forcing potential P is selected, the functional (20). The subscript of $\mathbf{J}$ identifies the number of independent parameters, as shown below.

On replacing (23) into (22), U may be expressed in terms of the independent fields as

$$U = \frac{1}{2}\int_V \begin{Bmatrix} \tilde{\sigma} \\ \tilde{\mathbf{e}} \\ \tilde{\mathbf{u}} \end{Bmatrix}^T \begin{bmatrix} j_{11}\mathbf{E}^{-1} & j_{12}\mathbf{I} & j_{13}\mathbf{D} \\ j_{12}\mathbf{I} & j_{22}\mathbf{E} & j_{23}\mathbf{ED} \\ j_{13}\mathbf{D}^T & j_{23}\mathbf{D}^T\mathbf{E} & j_{33}\mathbf{D}^T\mathbf{ED} \end{bmatrix} \begin{Bmatrix} \tilde{\sigma} \\ \tilde{\mathbf{e}} \\ \tilde{\mathbf{u}} \end{Bmatrix} \mathrm{d}V, \tag{27}$$

which verifies the symmetry of $\mathbf{J}_3$. Using (27) the first variation of U may be presented as

$$\delta U = (\Delta\mathbf{e}, \delta\tilde{\sigma})_V + (\Delta\sigma, \delta\tilde{\mathbf{e}})_V - (\mathbf{div}\,\sigma', \delta\tilde{\mathbf{u}})_V + [\sigma'_n, \delta\tilde{\mathbf{u}}]_S, \tag{28}$$

where

$$\Delta\mathbf{e} = j_{11}\mathbf{e}^\sigma + j_{12}\tilde{\mathbf{e}} + j_{13}\mathbf{e}^u, \quad \Delta\sigma = j_{12}\tilde{\sigma} + \mathrm{j}_{22}\sigma^e + \mathrm{j}_{23}\sigma^u,$$
$$\sigma' = j_{13}\tilde{\sigma} + \mathrm{j}_{23}\sigma^e + \mathrm{j}_{33}\sigma^u. \tag{29}$$

The last two terms in (28) combine with contributions from the forcing potential variation. For example, if P is the conventional forcing potential (21), the complete variation of $\Pi^c = U - P^c$ is

$$\delta\Pi^c = (\Delta\mathbf{e}, \delta\tilde{\sigma})_V + (\Delta\sigma, \delta\tilde{\mathbf{e}})_V - (\mathbf{div}\,\sigma' + \mathbf{b}, \delta\tilde{\mathbf{u}})_V + [\sigma'_n - \hat{\mathbf{t}}, \delta\tilde{\mathbf{u}}]_{S_t} - [\tilde{\mathbf{u}} - \hat{\mathbf{d}}, \delta\tilde{\sigma}_n]_{S_d}. \tag{30}$$

Using P^d or P^t does not change the volume terms. Consequently the Euler equations associated with the volume terms of the first variation

$$\Delta\mathbf{e} = 0, \quad \Delta\sigma = 0, \quad \mathbf{div}\,\sigma' + \mathbf{b} = 0, \tag{31}$$

are independent of the forcing potential.

For consistency of the Euler equations with the field equations (1)–(3), one must have $\Delta\mathbf{e} = \mathbf{0}$, $\Delta\sigma = 0$ and $\sigma' = \sigma$ if the assumed stress and strain fields reduce to the exact ones. Therefore

$$j_{11} + j_{12} + j_{13} = 0,$$
$$j_{12} + j_{22} + j_{23} = 0,$$
$$j_{13} + j_{23} + j_{33} = 1. \tag{32}$$

Because of these constraints, the maximum number of independent parameters that define the entries of $\mathbf{J}_3$ is three as claimed. The specialization of these functionals to conventional and parametrized forms is discussed in Felippa and Militello (1989, 1990).

5. SPLIT FORM OF GENERALIZED STRAIN ENERGY

The expression (22) for U is not suitable for incompressible materials. To construct a parametrized form that encompasses incompressibility the generalized strain energy is augmented with additional independent fields, one of which must be the pressure. There are several ways of accomplishing this objective. Here the starting point is the conventional deviatoric splitting (18) and construction of an augmented generalized strain energy U_{ds} (subscripts stand for "deviatoric split") in terms of the five independent fields $\tilde{\mathbf{s}}$, $\tilde{\mathbf{g}}$, $\tilde{\mathbf{u}}$, $\tilde{p}$ and $\tilde{\theta}$. Using (25) as a "template" the following quadratic form is postulated

$$U_{ds} = \frac{1}{2}\int_V \begin{Bmatrix} \tilde{\mathbf{s}} \\ \mathbf{s}^d \\ \mathbf{s}^u \\ \tilde{p} \\ p^d \\ p^u \end{Bmatrix}^T \begin{bmatrix} j_{11}\mathbf{I} & j_{12}\mathbf{I} & j_{13}\mathbf{I} & j_{14}\mathbf{h} & j_{15}\mathbf{h} & j_{16}\mathbf{h} \\ j_{21}\mathbf{I} & j_{22}\mathbf{I} & j_{23}\mathbf{I} & j_{24}\mathbf{h} & j_{25}\mathbf{h} & j_{26}\mathbf{h} \\ j_{31}\mathbf{I} & j_{32}\mathbf{I} & j_{33}\mathbf{I} & j_{34}\mathbf{h} & j_{35}\mathbf{h} & j_{36}\mathbf{h} \\ j_{41}\mathbf{h}^T & j_{42}\mathbf{h}^T & j_{43}\mathbf{h}^T & j_{44} & j_{45} & j_{46} \\ j_{51}\mathbf{h}^T & j_{52}\mathbf{h}^T & j_{53}\mathbf{h}^T & j_{54} & j_{55} & j_{56} \\ j_{61}\mathbf{h}^T & j_{62}\mathbf{h}^T & j_{63}\mathbf{h}^T & j_{64} & j_{65} & j_{66} \end{bmatrix} \begin{Bmatrix} \mathbf{g}^\sigma \\ \tilde{\mathbf{g}} \\ \mathbf{g}^u \\ \theta^p \\ \tilde{\theta} \\ \theta^u \end{Bmatrix} \mathrm{d}V, \tag{33}$$

in which the derived fields are

$$\mathbf{g}^u = (\mathbf{D} - \tfrac{1}{3}\mathbf{h}\,\mathbf{div})\tilde{\mathbf{u}} = \mathbf{D}_d\tilde{\mathbf{u}}, \quad \mathbf{g}^\sigma = \mathbf{C}^{-1}\tilde{\mathbf{s}}, \quad \theta^p = k^{-1}\tilde{p}, \quad \theta^u = -\mathbf{div}\,\tilde{\mathbf{u}},$$
$$\mathbf{s}^d = \mathbf{C}\tilde{\mathbf{g}}, \quad \mathbf{s}^u = \mathbf{C}\mathbf{g}^u = \mathbf{C}\mathbf{D}_d\tilde{\mathbf{u}}, \quad p^d = k\tilde{\theta}, \quad p^u = k\theta^u = -k\,\mathbf{div}\,\tilde{\mathbf{u}}. \tag{34}$$

The kernel matrix of the quadratic form (33) is now 21×21 and is characterized by the 36 j coefficients. Unlike the treatment in Section 4, coefficient symmetry conditions are not set *ab initio*. Substituting (34) into (33), U_{ds} may be expressed in terms of the five independent fields as the quadratic form

$$U_{ds} = \frac{1}{2}\int_V \begin{Bmatrix} \tilde{\mathbf{s}} \\ \tilde{\mathbf{g}} \\ \tilde{\mathbf{u}} \\ \tilde{p} \\ \tilde{\theta} \end{Bmatrix}^T \begin{bmatrix} j_{11}\mathbf{C}^{-1} & j_{12}\mathbf{I} & j_{13}\mathbf{D}_d + j_{16}\mathbf{h}\,\mathbf{div} & j_{14}k^{-1}\mathbf{h} & j_{15}\mathbf{h} \\ j_{21}\mathbf{I} & j_{22}\mathbf{C} & j_{23}\mathbf{C}\mathbf{D}_d + j_{26}\mathbf{C}\mathbf{h}\,\mathbf{div} & j_{24}k^{-1}\mathbf{C}\mathbf{h} & j_{25}\mathbf{C}\mathbf{h} \\ j_{31}\mathbf{D}_d^T + j_{61}k\,\mathbf{grad}\,\mathbf{h}^T\mathbf{C}^{-1} & j_{32}\mathbf{D}_d^T\mathbf{C} + j_{62}k\,\mathbf{grad}\,\mathbf{h}^T & \begin{matrix}\mathbf{D}_d^T\mathbf{C}(j_{33}\mathbf{D}_d + j_{36}\mathbf{h}\,\mathbf{div}) \\ + k\,\mathbf{grad}\,(j_{63}\mathbf{h}^T\mathbf{D}_d + j_{66}\,\mathbf{div})\end{matrix} & j_{34}k^{-1}\mathbf{D}_d^T\mathbf{C}\mathbf{h} + j_{64}\,\mathbf{grad} & j_{35}\mathbf{D}_d^T\mathbf{C}\mathbf{h} + j_{65}k\,\mathbf{grad} \\ j_{41}\mathbf{h}^T\mathbf{C}^{-1} & j_{42}\mathbf{h}^T & j_{43}\mathbf{h}^T\mathbf{D}_d + j_{46}\,\mathbf{div} & j_{44}k^{-1} & j_{45} \\ j_{51}k\mathbf{h}^T\mathbf{C}^{-1} & j_{52}k\mathbf{h}^T & j_{53}k\mathbf{h}^T\mathbf{D}_d + j_{56}k\,\mathbf{div} & j_{54} & j_{55}k \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{s}} \\ \tilde{\mathbf{g}} \\ \tilde{\mathbf{u}} \\ \tilde{p} \\ \tilde{\theta} \end{Bmatrix} \mathrm{d}V \tag{35}$$

in which

$$\mathbf{grad} \equiv \mathbf{div}^T = \{\partial/\partial x_1 \quad \partial/\partial x_2 \quad \partial/\partial x_3\}^T$$

when applied to a scalar function. The kernel matrix in (35) must be symmetric, a condition that provides the following symmetry relations:

$$j_{mn} = j_{nm}, \quad m = 1, 2, 3 \quad n = 1, 2, 3 \qquad j_{mn} = j_{nm}, \quad m = 4, 5, 6 \quad n = 4, 5, 6$$
$$j_{mn}\mathbf{I} = j_{nm}k^{-1}\mathbf{C}, \quad m = 4, 5, 6 \quad n = 1, 2, 3. \tag{36}$$

If these conditions are imposed on (33) that kernel matrix becomes

$$\begin{bmatrix} j_{11}\mathbf{I} & j_{12}\mathbf{I} & j_{13}\mathbf{I} & j_{14}\mathbf{h} & j_{15}\mathbf{h} & j_{16}\mathbf{h} \\ j_{12}\mathbf{I} & j_{22}\mathbf{I} & j_{23}\mathbf{I} & j_{24}\mathbf{h} & j_{25}\mathbf{h} & j_{26}\mathbf{h} \\ j_{13}\mathbf{I} & j_{32}\mathbf{I} & j_{33}\mathbf{I} & j_{34}\mathbf{h} & j_{35}\mathbf{h} & j_{36}\mathbf{h} \\ j_{14}k^{-1}\mathbf{Ch}^T & j_{24}k^{-1}\mathbf{Ch}^T & j_{34}k^{-1}\mathbf{Ch}^T & j_{44} & j_{45} & j_{46} \\ j_{15}k^{-1}\mathbf{Ch}^T & j_{25}k^{-1}\mathbf{Ch}^T & j_{35}k^{-1}\mathbf{Ch}^T & j_{45} & j_{55} & j_{56} \\ j_{16}k^{-1}\mathbf{Ch}^T & j_{26}k^{-1}\mathbf{Ch}^T & j_{36}k^{-1}\mathbf{Ch}^T & j_{46} & j_{56} & j_{66} \end{bmatrix}. \tag{37}$$

This is fully characterized by the 6×6 functional-generating symmetric matrix

$$\mathbf{J}_{12} = \begin{bmatrix} j_{11} & j_{12} & j_{13} & j_{14} & j_{15} & j_{16} \\ j_{12} & j_{22} & j_{23} & j_{24} & j_{25} & j_{26} \\ j_{13} & j_{23} & j_{33} & j_{34} & j_{35} & j_{36} \\ j_{14} & j_{24} & j_{34} & j_{44} & j_{45} & j_{46} \\ j_{15} & j_{25} & j_{35} & j_{45} & j_{55} & j_{56} \\ j_{16} & j_{26} & j_{36} & j_{46} & j_{56} & j_{66} \end{bmatrix} \tag{38}$$

(the $\mathbf{J}$ subscript denotes the number of free parameters, as explained below). The kernel matrix of (35) becomes

$$\begin{bmatrix} j_{11}\mathbf{C}^{-1} & j_{12}\mathbf{I} & j_{13}\mathbf{D}_q - j_{16}\mathbf{h}\,\mathbf{div} & j_{14}k^{-1}\mathbf{h} & j_{15}\mathbf{h} \\ & j_{22}\mathbf{C} & j_{23}\mathbf{CD}_q - j_{26}\mathbf{Ch}\,\mathbf{div} & j_{24}k^{-1}\mathbf{Ch} & j_{25}\mathbf{Ch} \\ & & j_{33}\mathbf{D}_q^T\mathbf{CD}_q + j_{66}k\,\mathbf{grad}\,\mathbf{div}) & j_{34}k^{-1}\mathbf{D}_q^T\mathbf{Ch} & j_{35}\mathbf{D}_q^T\mathbf{Ch} \\ & & -j_{36}(\mathbf{D}_q^T\mathbf{Ch}\,\mathbf{div} + \mathbf{grad}\,\mathbf{h}^T\mathbf{CD}_q) & -j_{46}\,\mathbf{grad} & -j_{56}k\,\mathbf{grad} \\ & & & j_{44}k^{-1} & j_{45} \\ \mathit{symm} & & & & j_{55}k \end{bmatrix}. \tag{39}$$

The first variation of (35) is

$$\delta U_{ds} = (\Delta\mathbf{g}, \delta\tilde{\mathbf{s}})_V + (\Delta\mathbf{s}, \delta\tilde{\mathbf{g}})_V - (\mathbf{div}\,\boldsymbol{\sigma}', \delta\tilde{\mathbf{u}})_V + (\Delta\theta, \delta\tilde{p})_V + (\Delta p, \delta\tilde{\theta})_V + [\boldsymbol{\sigma}'_n, \delta\tilde{\mathbf{u}}]_S, \tag{40}$$

where

$$\begin{aligned} \Delta\mathbf{g} &= j_{11}\mathbf{g}^{\epsilon} + j_{12}\tilde{\mathbf{g}} + j_{13}\mathbf{g}^u + \mathbf{h}(j_{14}\theta^p + j_{15}\tilde{\theta} + j_{16}\theta^u), \\ \Delta\mathbf{s} &= j_{12}\tilde{\mathbf{s}} + j_{22}\tilde{\mathbf{s}}^{\sigma} + j_{23}\mathbf{s}^u + \mathbf{Ch}(j_{24}\theta^p + j_{25}\tilde{\theta} + j_{26}\theta^u), \\ \boldsymbol{\sigma}' &= j_{13}\tilde{\mathbf{s}} + j_{23}\mathbf{s}^{\sigma} + j_{33}\mathbf{s}^u + \mathbf{B}(j_{34}\theta^p + j_{35}\theta + j_{36}\theta^u) \\ &\quad + \mathbf{hh}^T(j_{16}\tilde{\mathbf{s}} + j_{26}\mathbf{s}^{\sigma} + j_{36}\mathbf{s}^u) - \mathbf{h}(j_{46}p + j_{56}p^{\theta} + j_{66}p^u) \\ &= j_{13}\tilde{\mathbf{s}} + j_{23}\mathbf{s}^{\sigma} + j_{33}\mathbf{s}^u + \mathbf{B}(j_{34}\theta^p + j_{35}\theta + j_{36}\theta^u) - \mathbf{h}(j_{46}p + j_{56}p^{\theta} + j_{66}p^u), \\ \Delta\theta &= \mathbf{h}^Tk^{-1}(j_{14}\tilde{\mathbf{s}} + j_{24}\mathbf{s}^{\sigma} + j_{34}\mathbf{s}^u) + j_{44}\theta^p + j_{45}\tilde{\theta} + j_{46}\theta^u = j_{44}\theta^p + j_{45}\tilde{\theta} + j_{46}\theta^u, \\ \Delta p &= \mathbf{h}^T(j_{15}\tilde{\mathbf{s}} + j_{25}\mathbf{s}^{\sigma} + j_{35}\mathbf{s}^u) + j_{45}\tilde{p} + j_{55}p^{\theta} + j_{56}p^u = j_{45}\tilde{p} + j_{55}p^{\theta} + j_{56}p^u \end{aligned} \tag{41}$$

where $\mathbf{B} = (\mathbf{I} - \tfrac{1}{3}\mathbf{hh}^T)\mathbf{Ch}$, and the simplifications in $\boldsymbol{\sigma}'$, $\Delta\theta$ and Δp result from $\mathbf{h}^T\mathbf{s} = \mathbf{h}^T\mathbf{s}^{\sigma} = \mathbf{h}^T\mathbf{s}^u = 0$ since the deviatoric stress tensor is traceless. Applying again the consistency argument and noting that mean and deviatoric parts may vary independently, one obtains the constraint conditions

$$
\begin{aligned}
&j_{11}+j_{12}+j_{13} = 0, \quad j_{14}+j_{15}+j_{16} = 0, \quad j_{12}+j_{22}+j_{23} = 0,\\
&j_{24}+j_{25}+j_{26} = 0, \quad j_{13}+j_{23}+j_{33} = 1, \quad j_{34}+j_{35}+j_{36} = 0,\\
&j_{46}+j_{56}+j_{66} = 1, \quad j_{44}+j_{45}+j_{46} = 0, \quad j_{45}+j_{55}+j_{56} = 0.
\end{aligned}
\tag{42}
$$

Because of these nine constraints the maximum number of independent parameters that define the coefficients of matrix (38) is $21-9 = 12$ as claimed.

6. SIMPLIFICATIONS

Having a ∞^{12} family of functionals for constructing approximation methods such as finite elements leaves the selection wide open. In the absence of other information it appears prudent to reduce the number of free parameters by setting all coefficients that couple mean and deviatoric quantities equal to zero:

$$
\mathbf{J}_6 = \begin{bmatrix}
j_{11} & j_{12} & j_{13} & 0 & 0 & 0\\
j_{12} & j_{22} & j_{23} & 0 & 0 & 0\\
j_{13} & j_{32} & j_{33} & 0 & 0 & 0\\
0 & 0 & 0 & j_{44} & j_{45} & j_{46}\\
0 & 0 & 0 & j_{45} & j_{55} & j_{56}\\
0 & 0 & 0 & j_{46} & j_{56} & j_{66}
\end{bmatrix}
\tag{43}
$$

subject to the constraints that the row (and column) sums be 0, 0, 1, 0, 0 and 1 respectively. This simplified form exhibits six independent parameters.

The next question is how to include exact incompressibility, for which $k \to \infty$. A study of the matrix (39) reveals that the only coefficients affecting terms multiplied by k are j_{55} and j_{66}. One solution would be to take $j_{55} = j'_{55}/k$, and $j_{66} = j'_{66}/k$ with the primed coefficients as source data. A more expedient solution is to set those coefficients to zero, which reduces (43) to

$$
\mathbf{J}_4 = \begin{bmatrix}
j_{11} & j_{12} & j_{13} & 0 & 0 & 0\\
j_{12} & j_{22} & j_{23} & 0 & 0 & 0\\
j_{13} & j_{32} & j_{33} & 0 & 0 & 0\\
0 & 0 & 0 & 2\omega-1 & -\omega & 1-\omega\\
0 & 0 & 0 & -\omega & 0 & \omega\\
0 & 0 & 0 & 1-\omega & \omega & 0
\end{bmatrix}
\tag{44}
$$

where ω is a free parameter that determines the lower 3×3 principal minor. The total number of parameters is reduced to four, just one more than in compressible elasticity. Thus the following practical rule emerges: any compressible-elasticity principle characterized by the coefficients (26) can be extended to embody incompressibility by modifying U as follows:

(a) Replace $\boldsymbol{\sigma}$ and $\mathbf{e}$ by $\mathbf{s}$ and $\mathbf{g}$, respectively. (In fact, only the first modification is actually needed, since $\mathbf{s}^T\mathbf{g} = \mathbf{s}^T\mathbf{e}$, etc.)

(b) Add the pressure and volumetric strain terms characterized by the lower 3×3 principal minor in (44). If ω is zero the volumetric strain drops out as independent field and the additional terms reduce to

$$
\tfrac{1}{2}(\tilde{p}, \theta^u - \theta^p)_V + \tfrac{1}{2}(p^u, \theta^p)_V = -\int_V \left[\frac{\tilde{p}^2}{2k} + \tilde{p}\,\mathbf{div}\,\tilde{\mathbf{u}}\right] \mathrm{d}V.
\tag{45}
$$

Furthermore, in exact incompressibility only the term $-\tilde{p}\,\mathbf{div}\,\mathbf{u}$ survives.

7. LAME SPLITTING

Consideration of the Lamé splitting (19) is of interest because of historical reasons, since the first mixed principle encompassing compressible and incompressible elasticity constructed by Herrmann (1965) was based on it. Again one can start by postulating a quadratic form for the generalized strain energy U_{Ls} (where subscripts stand for "Lamé split"):

$$U_{Ls} = \frac{1}{2}\int_V \begin{Bmatrix} \tilde{\boldsymbol{\tau}} \\ \boldsymbol{\tau}^e \\ \boldsymbol{\tau}^u \\ \tilde{q} \\ q^\theta \\ q^u \end{Bmatrix}^T \begin{bmatrix} l_{11}\mathbf{I} & l_{12}\mathbf{I} & l_{13}\mathbf{I} & l_{14}\mathbf{h} & l_{15}\mathbf{h} & l_{16}\mathbf{h} \\ l_{21}\mathbf{I} & l_{22}\mathbf{I} & l_{23}\mathbf{I} & l_{24}\mathbf{h} & l_{25}\mathbf{h} & l_{26}\mathbf{h} \\ l_{31}\mathbf{I} & l_{32}\mathbf{I} & l_{33}\mathbf{I} & l_{34}\mathbf{h} & l_{35}\mathbf{h} & l_{36}\mathbf{h} \\ l_{41}\mathbf{h}^T & l_{42}\mathbf{h}^T & l_{43}\mathbf{h}^T & l_{44} & l_{45} & l_{46} \\ l_{51}\mathbf{h}^T & l_{52}\mathbf{h}^T & l_{53}\mathbf{h}^T & l_{54} & l_{55} & l_{56} \\ l_{61}\mathbf{h}^T & l_{62}\mathbf{h}^T & l_{63}\mathbf{h}^T & l_{64} & l_{65} & l_{66} \end{bmatrix} \begin{Bmatrix} \mathbf{e}^s \\ \tilde{\mathbf{e}} \\ \mathbf{e}^u \\ \theta^q \\ \tilde{\theta} \\ \theta^u \end{Bmatrix} \mathrm{d}V, \tag{46}$$

in which the l's coefficients take the place of the js, and where the new terms are

$$\begin{aligned} &\tilde{\boldsymbol{\tau}} = \boldsymbol{\sigma} - q\mathbf{h}, \quad \boldsymbol{\tau}^e = \mathbf{C}\tilde{\mathbf{e}}, \quad \boldsymbol{\tau}^u = \mathbf{CDu}, \quad \xi = 3\nu/(1+\nu), \\ &\tilde{q} = \xi\tilde{p}, \quad q^\theta = \xi\lambda\tilde{\theta}, \quad q^u = -\xi\lambda\,\mathbf{div}\,\tilde{\mathbf{u}}, \quad \theta^q = q/\lambda. \end{aligned} \tag{47}$$

Going through the same mechanics one obtains relations similar to (35)–(40) with s, g, p, k and $\mathbf{D}_g$ replaced by $\boldsymbol{\tau}$, e, q, λ and $\mathbf{D}$, respectively. But now $\mathbf{h}^T\boldsymbol{\tau}$ is not necessarily zero and consequently the counterpart of (41) retains more terms:

$$\begin{aligned} \Delta\mathbf{e} &= l_{11}\mathbf{e}^s + l_{12}\tilde{\mathbf{e}} + l_{13}\mathbf{e}^u + \mathbf{h}(l_{14}\theta^q + l_{15}\tilde{\theta} + l_{16}\theta^u), \\ \Delta\boldsymbol{\tau} &= l_{12}\tilde{\boldsymbol{\tau}} + l_{22}\tilde{\boldsymbol{\tau}}^e + l_{23}\boldsymbol{\tau}^u + \mathbf{Ch}(l_{24}\theta^q + l_{25}\tilde{\theta} + l_{26}\theta^u), \\ \boldsymbol{\sigma}' &= l_{13}\tilde{\boldsymbol{\tau}} + l_{23}\boldsymbol{\tau}^e + l_{33}\boldsymbol{\tau}^u + \mathbf{Ch}(l_{34}\theta^q + l_{35}\theta + l_{36}\theta^u) \\ &\quad + \mathbf{hh}^T(l_{16}\tilde{\boldsymbol{\tau}} + l_{26}\boldsymbol{\tau}^e + l_{36}\boldsymbol{\tau}^u) - \mathbf{h}(l_{46}p + l_{56}p^\theta + l_{66}p^u) \\ \Delta\theta &= \mathbf{h}^T\lambda^{-1}(l_{14}\tilde{\boldsymbol{\tau}} + l_{24}\boldsymbol{\tau}^e + l_{34}\boldsymbol{\tau}^u) + l_{44}\theta^q + l_{45}\tilde{\theta} + l_{46}\theta^u, \\ \Delta q &= \mathbf{h}^T(l_{15}\tilde{\boldsymbol{\tau}} + l_{25}\boldsymbol{\tau}^e + l_{35}\boldsymbol{\tau}^u) + l_{45}\tilde{q} + l_{55}q^\theta + l_{56}q^u. \end{aligned} \tag{48}$$

Consistency with the field equations provides the twelve constraints

$$\begin{aligned} &l_{11} + l_{12} + l_{13} = 0, \quad l_{14} + l_{15} + l_{16} = 0, \quad l_{12} + l_{22} + l_{23} = 0, \\ &l_{24} + l_{25} + l_{26} = 0, \quad l_{13} + l_{23} + l_{33} = 1, \quad l_{34} + l_{35} + l_{36} = 0, \\ &l_{16} + l_{26} + l_{36} = 0, \quad l_{46} + l_{56} + l_{66} = 1, \quad l_{14} + l_{24} + l_{34} = 0, \\ &l_{44} + l_{45} + l_{46} = 0, \quad l_{15} + l_{25} + l_{35} = 0, \quad l_{45} + l_{55} + l_{56} = 0. \end{aligned} \tag{49}$$

This leaves $21 - 12 = 9$ independent parameters in the functional-generating symmetric matrix

$$\mathbf{L}_9 = \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ l_{12} & l_{22} & l_{23} & l_{24} & l_{25} & l_{26} \\ l_{13} & l_{23} & l_{33} & l_{34} & l_{35} & l_{36} \\ l_{14} & l_{24} & l_{34} & l_{44} & l_{45} & l_{46} \\ l_{15} & l_{25} & l_{35} & l_{45} & l_{55} & l_{56} \\ l_{16} & l_{26} & l_{36} & l_{46} & l_{56} & l_{66} \end{bmatrix}. \tag{50}$$

If the off-diagonal blocks of this matrix are set to zero as in (43), $\mathbf{L}_9$ becomes $\mathbf{L}_6$ and the conditions on the remaining nonzero coefficients are identical to those of $\mathbf{J}_6$.

Treatment of the more general splitting (12) with $\eta \neq 0$ does not cause any particular difficulties. However, as splittings other than (18) do not accommodate anisotropic materials naturally, they will not be investigated further.

8. SPECIALIZATIONS

The simplest principle (in the sense of having the most sparse J matrix) that accommodates both compressible and incompressible elasticity is obtained by specializing (44) to

$$\mathbf{J}_P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{51}$$

This choice leaves only displacements and pressures as independent field variables and yields

$$U_P(\mathbf{u}, p) = \tfrac{1}{2}(\mathbf{s}^u, \mathbf{g}^u)_V - \left(\tilde{p}, \frac{\tilde{p}}{2k} + \mathbf{div}\,\tilde{\mathbf{u}}\right)_V = \tfrac{1}{2}(\mathbf{s}^u, \mathbf{e}^u)_V - \left(\frac{\tilde{p}^2}{2k} + \tilde{p}\,\mathbf{div}\,\tilde{\mathbf{u}}\right)_V, \tag{52}$$

which may be viewed as a modification of the minimum potential energy functional. For practical use it is important to note that $\mathbf{g}^u$ may be replaced by $\mathbf{e}^u$ in the first integral since tensor s^u_{ij} is traceless. In the incompressible limit U_P collapses to $\tfrac{1}{2}(\mathbf{s}^u, \mathbf{e}^u)_V - (\tilde{p}, \mathbf{div}\,\mathbf{u})_V$.

The specialization

$$\mathbf{J}_{AR} = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{53}$$

reduces $U_{ds} - P$ to the five-field functional presented by Atluri and Reissner (1989; in that paper p and θ are defined as the negatives of the quantities used here). Notice that since both 3×3 principal minors of $\mathbf{J}_{AR}$ display the Hu–Washizu structure of compressible elasticity, use of (24) yields

$$U_{AR} = U_H(\tilde{\mathbf{s}}, \tilde{\mathbf{g}}, \tilde{\mathbf{u}}) + U_H(\tilde{p}\mathbf{h}, \tilde{\theta}\mathbf{h}, \theta^u\mathbf{h}) = \tfrac{1}{2}(\mathbf{s}^g, \tilde{\mathbf{g}})_V + (\tilde{\mathbf{s}}, \mathbf{g}^u - \tilde{\mathbf{g}})_V + \tfrac{1}{2}(p^\theta, \tilde{\theta})_V + \tilde{p}(\theta^u - \tilde{\theta})_V, \tag{54}$$

in which again $\mathbf{g}^u$ and $\tilde{\mathbf{g}}$ may be replaced by $\mathbf{e}^u$ and $\tilde{\mathbf{e}}$, respectively. As $j_{55} \neq 0$, this functional does not accommodate exact incompressibility. This drawback can be easily corrected, however, through the techniques discussed in Section 6.

Finally, specialization of (50) to

$$\mathbf{L}_H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{L}_F = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{55}$$

reduces the functional $U_{L_s} - P$ to those presented by Herrmann (1965) and Franca (1989), respectively; which are identified as $U_H - P$ and $U_F - P$ in the sequel.

Herrmann's functional, which as noted above has historical importance, contains two independent fields: displacements **u** and pseudo pressure q. Its U functional is

$$U_H(\tilde{\mathbf{u}}, \tilde{q}) = \tfrac{1}{2}(\boldsymbol{\tau}^u, \mathbf{e}^u)_V - \left(\frac{\tilde{q}^2}{2\lambda} + \tilde{q}\,\mathrm{div}\,\tilde{\mathbf{u}}\right)_V. \tag{56}$$

The upper and lower 3×3 principal minors of $\mathbf{L}_H$ display the structure of the minimum potential energy and stress–displacement Reissner compressible elasticity functions, respectively.

Franca's functional contains four independent fields: extra stress $\boldsymbol{\tau}$, total strains **e**, displacements **u** and pseudo pressure q. Its U functional is

$$U_F(\tilde{\boldsymbol{\tau}}, \tilde{\mathbf{e}}, \tilde{\mathbf{u}}, q) = \tfrac{1}{2}(\boldsymbol{\tau}^d, \tilde{\mathbf{e}})_V + (\tilde{\boldsymbol{\tau}}, \mathbf{e}^u - \tilde{\mathbf{e}})_V - \left(\frac{\tilde{q}^2}{2\lambda} + \tilde{q}\,\mathrm{div}\,\tilde{\mathbf{u}}\right)_V. \tag{57}$$

The upper and lower 3×3 principal minors of $\mathbf{L}_F$ display the structure of the Hu–Washizu and stress-displacement Reissner compressible-elasticity functions, respectively.

9. CONCLUSIONS

The parametrized formulations presented here extend the parametrized functionals of Felippa and Militello (1989, 1990) to accommodate incompressibility. In doing so a wider and perhaps bewildering range of possibilities is encountered, which raises some questions as regards the usefulness of functional parametrization techniques.

The formulation of parametrized variational principles offers conceptual and practical advantages. From a conceptual standpoint the technique is intellectually satisfying in that all possible variational forms are obtained once and for all. This should be contrasted to the conventional case-by-case derivation, which can only take "potshots" at the infinite domain of possible functionals. The key practical advantage is that generating matrix coefficients may be left free in finite element applications down to the element level, and used to enhance the quality of the numerical approximations as discussed in Felippa (1989a,b,c) and Felippa and Militello (1989, 1990).

However, coming face to face with twelve free parameters as in Section 5 may be confusing and negate the claimed benefits of generality. The simplifications of Section 6 appear reasonable from an applications standpoint because: (1) they cut the number of independent parameters while retaining flexibility in the weighting of the participating fields, and (2) all important specific functionals proposed to date are still covered.

Finally, the simplicity and generality of the functionals based on the deviatoric splitting (18) should be kept in mind. It is difficult to understand why the finite element literature is still preoccupied with the Lamé splitting and associated functionals. Not only is this splitting unnatural for anisotropic materials but note that associated functionals such as (56) and (57) degenerate for $\lambda = 0$, which happens if $\nu = 0$. At this value, $\xi = 0$, q vanishes identically, and 0/0 terms requiring special treatment appear in U. As a zero Poisson's ratio is physically realizable the claim to generality of application, even with restriction to isotropic behavior, is seriously weakened.

Acknowledgements—The work has been supported by NASA Lewis Research Center under Grant NAG 3-934.

REFERENCES

Atluri, S. N. and Reissner, E. (1989). On the formulation of variational theorems involving volume constraints. *Comput. Mech.* **5**, 337–344.

Felippa, C. A. (1989a). Parameterized multifield variational principles in elasticity: I. Mixed functionals. *Commun. Appl. Numer. Meth.* **5**, 79–88.

Felippa, C. A. (1989b). Parameterized multifield variational principles in elasticity: II. Hybrid functionals and the free formulation. *Commun. Appl. Numer. Meth.* **5**, 89–98.

Felippa, C. A. (1989c). The extended free formulation of finite elements in linear elasticity. *J. Appl. Mech.* **56**, 609–616.

Felippa, C. A. and Militello, C. (1989). Developments in variational methods for high-performance plate and shell elements. In *Analytical and Computational Methods for Shells* (Edited by A. K. Noor, T. Belytschko and J. C. Simo), CAD Vol. 3, pp. 191–216. ASME, New York.

Felippa, C. A. and Militello, C. (1990). The variational formulation of high-performance finite elements: parameterized variational principles. *Comput. Struct.* **36**, 1–11.

Franca, L. P. (1989). Analysis and finite element approximation of compressible and incompressible linear isotropic elasticity based upon a variational principle. *Comput. Meth. Appl. Mech. Engng* **76**, 259–273.

Gurtin, M. (1972). The linear theory of elasticity. In *Encyclopedia of Physics* (Edited by C. Truesdell), Vol. VIa/2, pp. 1–295. Springer, Berlin.

Herrmann, L. R. (1965). Elasticity equations for nearly incompressible materials by a variational theorem. *AIAA J.* **3**, 1896–1900.